Canadian Journal of Pure and Applied Sciences
Vol. 9, No. 1, pp. 3307-3311, February 2015
Online ISSN: 1920-3853; Print ISSN: 1715-9997
Available online at www.cjpas.net

DETERMINATION OF RELATIVE PREMIUM IN A BONUS-MALUS SYSTEM UNDER ORDER CONDITION

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ABSTRACT

The problem of the determination of the relative premiums of a L level Bonus-Malus system while the risk parameters in the system satisfy $\theta_1 < \theta_2 < ... < \theta_L$ is considered in this paper. In this paper used an MCMC Bayasian approach to estimating such constrained relative parameters. Also, several applications where estimation of such constrained relative parameters are needed have been given for some Bonus-Malus systems.

Keywords: Bonus-Malus system, constrained parameter space, relative premium, bayesian estimation, gibbs sampling.

INTRODUCTION

In many countries insurers use Bonus-Malus system in motor insurance in order to relate premium amounts based on policyholders experiance. Such a system penalize insured drivers responsible for at least one accidents (malus) and reward claim-free drivers(bonuses). In practice, a Bonus-Malus system consists of a finite number of levels, numbered from 1 to L, as policyholders classified according to their risk-tendency. In fact, the policyholder have the smallest risk-tendency stand the first level and pays the smallest premium. In the same manner, the policyholder have the largest risk-tendency stand the last level and pays the largest premium.

To reflect the heterogeneity in every level, the tendency to accident of each policyholder in the level l is characterized by a parameter that is called risk parameter. Risk parameter of policyholders in the level l, for $l = 1,..,L$, is denoted by θ_l. It is evident that in the Bonus-malus systems $\theta_1 < \theta_2 < ... < \theta_L$.

The estimators of risk parameters are called relative premiums and the relative premium associated to the level l is denoted as r_l. It means that the premium charged to a policyholder occupying the level l in the Bonus-Malus system has to pay r_l times the base premium to be covered by the insurance company (Denuit *et al*., 2007).

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If the value of relative premium of some level is lower than 1, i.e $r_l < 1$, it means that the policyholders in that level get a bonus. In the same manner, if the value of relative premium of a level is larger than 1, i.e $r_l > 1$, it means that the policyholders in that level get a malus. Suppose the maximum discount rate and the maximum surcharge rate of Bonus-Malus sytems,that are called super bonus and super malus, respectively, are a and b. So

$$a < r_1 < r_2 < ... < r_L < b. \tag{1}$$

Using quadratic loss function, the relative for level l is equals to (Norberg, 1976)

$$r_l = E(\Theta \mid \mathrm{L} = l) = \frac{\int_0^{+\infty} \theta \pi_l(\lambda\theta) d\pi_\Theta(\theta)}{\int_0^{+\infty} \pi_l(\lambda\theta) d\pi_\Theta(\theta)}, \quad 1 = 1,2,...,s.\cdot$$

This estimator may exhibit a rather irregular pattern, and this may be undesirable for commercial purposes (Denuit *et al*., 2007).

One method for estimating the risk parameters under order restriction is the use of isotonic regression of maximum likelihood. For details see Robertson *et al*. (1988) which contains much of the work related to statistical inference under order restrictions. Several studies have attempted to estimation problems in restricted or truncated parameter spaces. Marchand and Strawderman (2004) provide a review of estimation problems in restricted parameter spaces. Within this framework, many authors such as Kubokawa (1994, 2005a,b), Marchand and Payandeh (2011) dealt with Bayesian estimators of bounded location or scale parameters. A very nice feature is the use of MCMC Bayesian approach in which neither the method for estimating nor the computations become complicated as the dimension, L, increases. Broffitt (1984) carried out Bayesian calculations for constrained parameter and truncated data problems by means of the Gibbs sampler. He obtained the order-restricted Bayes estimator using inverse cumulative distribution function of conditional posterior distribution. Our specific procedure is the calculation of relative premiums under order restriction (1) based on Bayesian approach. Using Gibbs sampling, we develop alternative methods to estimate risk parameter vector Θ, i.e. calculation relative premiums, in the Bonus-Malus systems under order restriction (1).

The classical choices for prior distribution of Θ is the gamma distribution but there is no reason to restrict ourselves to this distribution. In fact, any distribution with support in the half positive real line is a good candidate to model the stochastic behavior of Θ. Inverse gaussian and log normal distributions is considered as π_Θ and the relative premiums under order restriction (1) is estimated. The parameters of π_Θ are chosen such that the condition of financial equilibrium is true. This means that, we expect that $E(\Theta)=1_L$. This ensures that the number of claims with and without considering risk parameter is very much the same.

This paper is developed as the following. The first section includes a review of the Gibbs sampling for estimating under order restriction. In next section a Bonus-Malus system is simulated and the risk parameters is estimated using three different prior distributions for Θ. While the results are given in final section.

The Gibbs Sampling

The Gibbs sampling was introduced by Geman and Geman (1984) in the context of image processing. Later, it was proposed as a general method for Bayesian calculation by Gelfand *et al*. (1992).

Let the conditionally independent observation $N_l, l=1,..,L$, the total observed claim numbers for a policyholder in the level l in last period, be available from distribution $f_N(n\,|\,\lambda,\Theta)$ is available from

$$f_N(n\,|\,\lambda,\theta)=\prod_{l=1}^{L}P(N_l=n_l\,|\,\lambda,\theta_l)=\prod_{l=1}^{L}\frac{\exp(-\lambda\theta_l)(\lambda\theta_l)^{n_l}}{n_l!};n_l=0,1,2,..., \tag{2}$$

where λ is a known function of the exposure-to-risk and possibly other covariates and N_l, $l=1,2,...,L$, is the claim number of a policyholder in the level l. For a prior distribution as $\pi(\Theta\,|\,\mu)$ for $\Theta=(\theta_1,...,\theta_L)$, the posterior distribution $\Theta=(\theta_1,...,\theta_L)$ given by $N=(N_1,...,N_L)$ is $\pi(\Theta\,|\,N,\mu)$. Gelfand *et al*. (1992) implemented the Gibbs sampling for ordered parameter applying $\Theta_l=\{\theta_k, k\neq l\}$ as the cross-section of Θ by the constraints on component θ_l at a specified set of values $\theta_k, k\neq l$. Using the method, the sampling is reduced to interval-restricted sampling from a standard distribution

$$\pi(\theta_l\,|\,\lambda,N,\mu,\Theta_j, j\neq l)\propto f_N(n\,|\,\lambda,\Theta)\pi(\Theta\,|\,\mu),$$

where $\theta_l\in\Theta_l=\{\theta_k,k\neq l\}$ and the right side is regarded as a function of θ_l for specified $\theta_j, j\neq l$.

Using this method, they implemented the Gibbs sampling by considering Θ in univariate cross-sections. In the following, Gibbs sampling under order restriction is reviewed.

Algorithm: Generate a sample size m of distribution (3) under order restriction (1) using Gibbs sampling by the following steps:

1. let $i=0$, and $\theta^{(0)}=(\theta_1^{(0)},...,\theta_L^{(0)})$ be initial values for the risk parameter vector, Θ.
2. For all $i<m$ repeat:
   - let $i=i+1$
   - generate a sample of distribution (3) by

$$\theta_1^{(i)}:\ \pi(\theta\,|\,\mu,\lambda,N_1,a,\theta_2^{(i-1)},...,\theta_s^{(i-1)})I_{(a,\theta_2^{(i-1)})}(\theta)$$

$$\theta_2^{(i)}:\ \pi(\theta\,|\,\mu,\lambda,N_2,\theta_1^{(i)},\theta_3^{(i-1)},...,\theta_s^{(i-1)})I_{(\theta_1^{(1)},\theta_2^{(i-1)})}(\theta)$$

$$\vdots$$

$$\theta_L^{(i)}:\ \pi(\theta\,|\,\mu,\lambda,N_L,\theta_1^{(i)},\theta_2^{(i)},...,\theta_{L-1}^{(i)},b)I_{(\theta_{L-1}^{(i)},b)}(\theta)$$

For the purpose of assessing convergence, we have used Gelman and Rubin diagnostic test (Gelman and Rubin, 1992). Their method recommends that two or more parallel chains be generated, each with different starting values. For assessing convergence of individual model parameters, the diagnostic test referred to as the potential scale reduction factor (PSRF), Gamerman and Lopes (2006). As chains converge to a common target distribution, the between-chain variability should become small relative to the within-chain variability and yields a PSRF that is close to 1. Conversely, PSRF values larger than 1 indicate non-convergence.

The relative premium of the level l, $l=1,2,...,L$, can be obtained as a sample estimate based on the $\theta_{lj}^{(T)}$ or possibly as a "Rao-Blakwellized" argument based on $\hat{E}(\theta_l\,|\,\mu,\lambda,N_l,\theta_1,\theta_2,...,\theta_L)=\frac{\sum_{i=1}^{m}\theta_l^{(i)}}{m}$.

If the posterior distribution is not proper, an option for sampling from the posterior is acceptance-rejection method.

In the next section a Bonus-Malus system is considered and relative premiums of policyholders is computed where the prior distribution of the risk parameter vector is a multivariate gamma, inverse gaussian and log normal under order restriction (1).

Simulation of a Bonus-Malus System

The Bonus-Malus system investigated here is assumed to possess 7 levels, labeled 1 to 7, and the distribution of the number of claims is (2) as $\lambda = 0.2$. The supper bonus and supper malus are $a = 0.3$, $b = 3$, respectively. Let the observed claim number of policyholders in the system during last period be $N = c(0,1,1,2,3,4,5)$. The aim is to determine relative premiums under order restriction (1). Also, for comparision, the relative premiums are estimated without order restriction, i.e. common estimator. In the following multivariate restricted gamma, inverse gaussian and log normal are used as prior distribution of Θ and the relative premiums are determined.

The Relative premiums under Restricted Gamma Prior

Suppose the prior distribution of the vector of risk parameter Θ under order restriction is

$$\pi(\Theta) = d_L(\delta_1,...,\delta_L;\gamma_1,...,\gamma_L)\prod_{l=1}^{L}\frac{\theta_l^{\delta_l-1}\exp(-\frac{\theta_l}{\gamma_l})}{\gamma_l^{\delta_l}\Gamma(\delta_l)}I(\theta_{l-1},\theta_{l+1}), \quad (4)$$

where d_L is the normalizing constant, δ_l,γ_l are parameters of a multivariate restricted gamma distribution and $\theta_0 = a, \theta_{L+1} = b$. Note that if θ_l's were unconstrained, (4) becomes a product of independent gamma priors. The joint posterior $\theta \mid N_l$ has the same form as (4) but δ_l replaced by ${\delta^*}_l = \delta_l + N_l$ and γ_l replaced by ${\gamma^*}_l = \dfrac{1}{\gamma_l + \lambda_l}$, i.e.

$$\pi(\Theta \mid \mu,\lambda,N,{\delta^*}_1,...,{\delta^*}_L;{\gamma^*}_1,...,{\gamma^*}_L) = d_L^*({\delta^*}_1,...,{\delta^*}_L;{\gamma^*}_1,...,{\gamma^*}_L)\prod_{l=1}^{L}\frac{\theta_l^{{\delta^*}_l-1}\exp(-\frac{\theta_l}{{\gamma^*}_l})}{{\gamma^*}_l^{{\delta^*}_l}\Gamma({\delta^*}_l)}I(\theta_{l-1},\theta_{l+1}), \quad (5)$$

Let the prior distribution of risk parameter of the level l, $l = 1,2,...,L$, be (4) with parameters $\delta_l = 1$ and $\gamma_l = 1$. The posterior distribution will be (5) where $\delta_l^* = 1 + N_l$ and $\gamma_l^* = \dfrac{1}{1.02}$. We ran the algorithms for 100,000 iteration after a burn-in of 10,000 iterations. The estimatod relative premiums is given in Table 1.

The Relative premiums under Restricted Inverse Gaussian Prior

Suppose the prior distribution of the vector of risk parameter Θ under order restriction is

$$\pi(\Theta) = d_L(\mu_1,...,\mu_L;\beta_1,...,\beta_L)\prod_{l=1}^{L}\frac{\mu_l}{\sqrt{2\pi\beta_l\theta_l^3}}\exp(-\frac{(\theta_l-\mu_l)^2}{2\beta_l\theta_l})I(\theta_{l-1},\theta_{l+1}),$$

where d_L is the normalizing constant, μ_l,β_l are parameters of a multivariate restricted inverse gaussian distribution and $\theta_0 = a, \theta_{L+1} = b$. Note that if θ_l's were unconstrained, (6) becomes a product of independent inverse gaussian priors. The joint posterior is

$$\pi(\Theta \mid \lambda,N,\mu_1,...,\mu_L,\beta_1,...,\beta_L) \propto \prod_{l=1}^{L}\theta_l^{N_l-3/2}\exp(-\frac{\theta_l^2(1+0.4\beta_l)-1}{2\theta_l})I(\theta_{l-1},\theta_{l+1}).$$

Since the full conditional distribution cannot be obtained easily, acceptance-rejection algorithm is used for sampling from the posterior distribution.

Now, let the prior distribution of risk parameter of the level l, $l = 1,2,...,L$, be (6) with parameters $\mu_l = 1$ and $\beta_l = 1$. The conditional posterior distribution of risk parameter of policyholders in the level l will be

$$\pi(\theta \mid \mu_l,\sigma_l,\lambda,N_l,\Theta_j; j \neq l) \propto \theta^{N_l-3/2}\exp(-\frac{1.4\theta^2-1}{2\theta})I(\theta_{l-1},\theta_{l+1}).$$

For acceptance-rejection algorithm, we need a proposal (or instrumental) distribution $q(.|.)$ which is χ^2 with parameters $\tilde{\mu}_l = \dfrac{1}{0.4}$ and $\sigma_l = 1 + N_l$, i.e.

$$q(\theta_l \mid \tilde{\mu}_l,\sigma_l,\theta_{l-1},\theta_{l+1}) = \chi^2(\tilde{\mu}_l,\sigma_l)I(\theta_{l-1},\theta_{L+1}).$$

This proposal distribution majorizes the posterior distribution (7) where $C = 5^{(N_l+1)}$.

We ran the algorithms for 100,000 iteration after a burn-in of 10,000 iterations. The estimatod relative premiums is given in table 2.

The Relative premiums under Restricted Log Nnormal Prior

Suppose the prior distribution of the vector of risk parameter Θ under order restriction is

$$\pi(\Theta) = d_L(\mu_1,...,\mu_L;\sigma_1,...,\sigma_L)\prod_{l=1}^{L}\frac{1}{\sqrt{2\pi}\theta_l\sigma_l}\exp(-\frac{(ln(\theta_l)-\mu_l)^2}{2\sigma_l^2})I(\theta_{l-1},\theta_{l+1}), \quad (8)$$

where d_L is the normalizing constant, μ_l,σ_l are parameters of multivariate restricted log normal distribution and $\theta_0 = a, \theta_{L+1} = b$. Note that if θ_l's were unconstrained, (8) becomes a product of independent Log normal priors. The joint posterior is

$$\pi(\Theta \mid \lambda,N,\mu_1,...,\mu_L,\beta_1,...,\beta_L) \propto \prod_{l=1}^{L}\exp(-\lambda\theta_l)\exp(-\frac{(ln(\theta_l)-\mu_l^*)^2}{2\sigma_l^2})I(\theta_{l-1},\theta_{l+1}),$$

Table 1. The results of calculated relative premiums under order restricted gamma prior where supper bonus and supper malus are 0.3 and 3, respectively.

Level / Relative premiums	L=1	L=2	L=3	L=4	L=5	L=6	L=7
Without order restriction	0.11	1.00	2.44	3.68	4.63	6.12	7.24
Under order restriction	0.49	0.84	1.31	1.78	2.19	2.52	2.78

Table 2. The results of calculated relative premiums under order restricted inverse gaussian prior where supper bonus and supper malus are 0.3 and 3, respectively.

Level / Relative premiums	L=1	L=2	L=3	L=4	L=5	L=6	L=7
Without order restriction	2.26	4.67	4.20	6.69	8.12	9.75	12.62
Under order restriction	0.80	1.30	1.71	2.08	2.39	2.64	2.84

Table 3. The results of calculated relative premiums under order restricted log normal prior where supper bonus and supper malus are 0.3 and 3, respectively.

Level / Relative premiums	L=1	L=2	L=3	L=4	L=5	L=6	L=7
Without order restriction	0.83	1.23	1.22	1.95	2.60	3.14	3.73
Under order restriction	1.05	1.43	1.74	2.04	2.32	2.57	2.80

where $\mu^*_l = \sigma_l^2(N_l + 0.5)$. Since the full conditional distribution cannot be obtained easily, acceptance-rejection algorithm is used for sampling from the posterior distribution.

Now, let the prior distribution of risk parameter of the level l, $l = 1,2,...,L$, be (8) with parameters $\mu_l = -0.125$ and $\sigma_l = 0.5$. The conditional posterior distribution of risk parameter of policyholders in the level l will be

$$\pi(\theta \mid \mu_l, \sigma_l, \lambda, N_l, \Theta_j; j \neq l) \propto \exp(-0.2\theta)\exp(-\frac{(ln(\theta)-(N_l+0.5))^2}{2})I(\theta_{l-1}, \theta_{l+1}). \quad (10)$$

For acceptance-rejection algorithm, we need a proposal (or instrumental) distribution $q(.|.)$ which is log normal distribution with parameters $\tilde{\mu}_l = 0.25(1.5 + N_l) + 0.6$ and $\sigma_l = 0.5$, i.e.

$q(\theta_l | \tilde{\mu}_l, \sigma_l, \theta_{l-1}, \theta_{l+1}) : \ Lognormal(\tilde{\mu}_l, \sigma_l) I(\theta_{l-1}, \theta_{L+1})$.

This proposal distribution majorizes the posterior distribution (10) where $C = 5^{(N_l+1)}$.

We ran the algorithms for 100,000 iteration after a burn-in of 10,000 iterations. The estimatod relative premiums is given in table 3.

Our specific procedure was the considering a new approach for adding risk parameter in a Bonus-Malus system. Also, the relative premiums under logical order restriction (1) calculated based on a Bayesian approach using Gibbs sampling. According obtained results in tables 1, 2 and 3, determined relative premiums without order restriction, the common method, are not reliable. The estimated relative premiums using Gibbs Sampling under constrainted parametre space are closely to actual values.

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Received: July 31, 2014; Revised: Sept 23, 2014;
Accepted: Oct 27, 2014